Distribution Centric Approach to Correlated Multi-Armed Bandits

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Overview

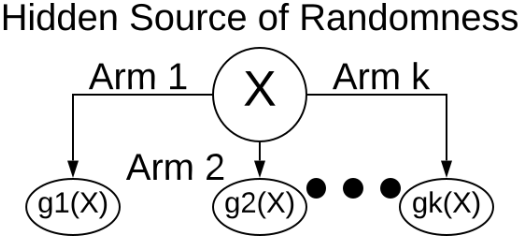
1. Introduction
2. Regret Minimization for Correlated Bandits
3. PAC Algorithm for Correlated Bandits

Introduction : What are Correlated Bandits?

- The usual MAB Setup has independent arms
- Independence assumption between arms is relaxed
- Correlation between arms can be exploited if present
- Skip pulling some arms based on correlation

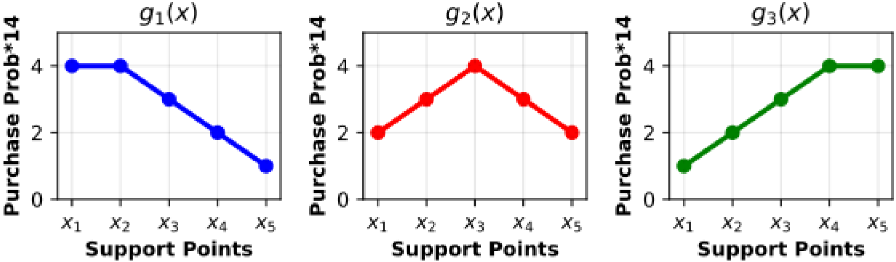
Introduction : What are correlated bandits?

- X is a discrete latent random variable
- Distribution of X is unknown
- $g_1(X), g_2(X), \ldots, g_k(X)$ are the dependent reward functions
- Here $g_1, g_2, \ldots, g_k$ are known functions



Introduction : An Example

An example of a situation where this model could be useful.
In general the arm functions are non-invertible.



Introduction : Overview of Reference Work

- Work of (Gupta et al. 2020)[2] describes a systematic method to exclude bad arms - CUCB
- Uses Distribution Agnostic Side information gathered for arm l using the pulls of arm k
- Skips sampling arms based on pairwise comparisons
- Excluded arms called *non competitive arms*
- To determine the arms to be excluded comparisons happen with certain reference arms
- Can we have a method that learns the underlying distribution and does more comparisons instead of just comparisons with a reference arm?

The UCUCB Algorithm

- Underlying distribution of X is learnt and used
- Called Pseudo distribution, an empirical estimate of the unknown distribution
- Computed as follows,

$$\tilde{p}_i(t) = \sum_{\tau=1}^{t} \frac{\beta_i(\tau)}{t} \tag{1}$$

Where, $\beta_i(\tau)$ is given by,

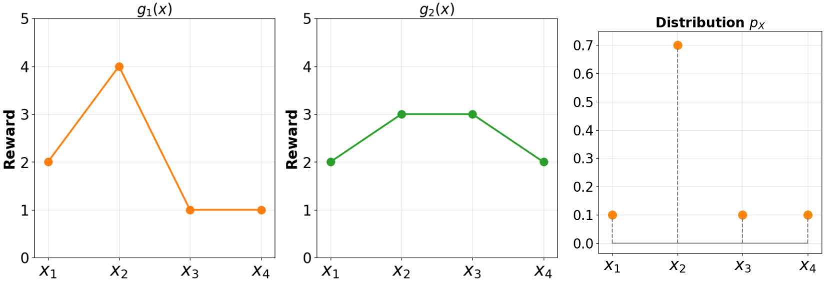
$$\beta_i(\tau) = \begin{cases} \frac{1}{|\text{inv}_k(r_\tau)|} & i \in \text{inv}_k(r_\tau) \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

Here $\text{inv}_k(r_\tau)$ is the preimage of the reward r_τ under the function g_k and $|\text{inv}_k(r)|$ is the cardinality of the set $\text{inv}_k(r_\tau)$

Problems with UCUCB

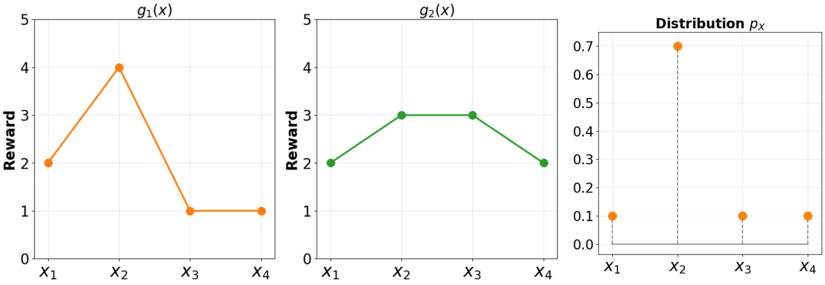
- UCUCB based on conjecture that persistent bias introduced into the distribution can only be due to indistinguishable points
- Uses a clause based on global pseudo distribution to remove non competitive arms
- Approach has limitations since it uses a global information estimate
- The idea of reference arms from CUCB is useful, makes comparison between performance of arms possible
- A global reference estimate such as the pseudo distribution cannot be relied upon for comparisons
- Pairwise distribution estimates do not make sense since they conceal all information

Problems: A Counter Example



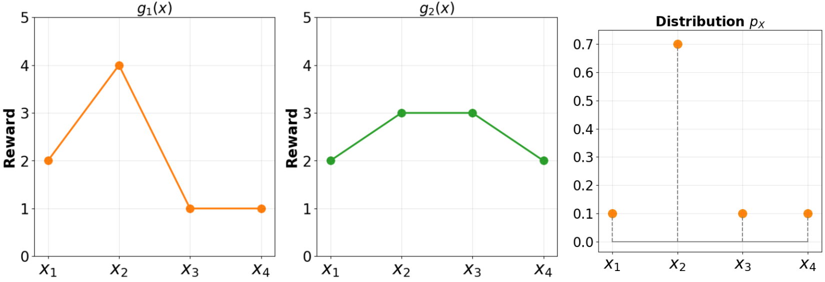
- Arm 2 pulled first, assume reward of 3 obtained
- As per the update rule we get $\tilde{P}_X(2) = [0, 0.5, 0.5, 0]$ for use in the next step
- This skewed distribution would make algorithm believe arm 2 is better

Problems: A Counter Example



- Now when arm 2 is sampled again and again the probability masses of positions 1 and 4 will remain in the neighbourhood of $\frac{1-(0.7+0.1)}{2} = 0.1$
- And through samples of arm 2, x_2 and x_3 will remain indistinguishable and their individual probability masses will never exceed 0.5
- $\tilde{P}(X)$ will always lie between $[0, 0.5, 0.5, 0]$ and $[0.1, 0.4, 0.4, 0.1]$. These distributions and everything in between will consider arm 2 to be superior to arm 1

Problems: A Counter Example



$E(t)$: The event that the algorithm breaks ties by pulling arm 2 first and obtains 3 as the reward

$$\sum_{t=1}^{T} \mathbb{E}[R(t)] = \sum_{t=1}^{T} \mathbb{E}[R(t)|E(t)]\mathbb{P}(E(t)) + \sum_{t=1}^{T} \mathbb{E}[R(t)|E^c(t)]\mathbb{P}(E^c(t)) \quad (3)$$

The first term on the RHS contributes linear regret. The second term is expected regret when $E(t)$ does not occur, which would still be non-negative.

Restrictions to Bandit Framework

- From counter example, it is clear that any temporary bias in the distribution is problematic
- Restriction that all arms be invertible is required
- Problem is no longer a partial observability Bandit Problem
- Becomes similar to experts setting with the restriction of drawing from a distribution
- Under this setting, constant cumulative expected regret can be achieved

Regret Minimization with Distribution (RMD)

Algorithm 1 RMD Algorithm

Input: Alphabet $\{x_1, ..., x_n\}$, Functions $\{g_1, ..., g_K\}$, All Invertible
Initialize : $t = 0, \tilde{g}_k = \infty$ (like Vanilla UCB)
for Every round t **do**
  $\tilde{g}_k(t) \leftarrow \sum_{i=1}^{n} g_k(x_i)\tilde{p}_i(t)$
  $k_t = \arg\max_k \tilde{g}_k(t)$
  Receive reward r_t by sampling arm k_t
  Record the realization of x, $x \leftarrow g_k^{-1}(r_t)$
  $t \leftarrow t + 1$
  $\tilde{p}_i(t+1) \leftarrow (\tilde{p}_i(t) \times t + \mathbb{1}_{x=x_i})/t$
end for

Analysis: Regret Upper Bound

Lemma (Hoeffding Inequality)

For a random variable $X \in (a, b)$,

$$\mathbb{P}\left(\frac{\sum_{\tau=1}^{t} X_\tau}{t} - \mu \geq \epsilon\right) \leq \exp\left(\frac{-2t\epsilon^2}{(b-a)^2}\right) \tag{4}$$

Applying the Hoeffding Inequality and using the fact that rewards are bounded between 0 and B, we have

Lemma (Number of Sub Optimal pulls)

The expected number of sub optimal pulls are bounded by,

$$\mathbb{E}\left[\sum_{\tau=1}^{t} \mathbb{1}_{k_\tau \neq k^*}\right] \leq K \sum_{\tau=1}^{t} \exp\left(\frac{-\tau \Delta_{\min}^2}{2B^2}\right) \tag{5}$$

Analysis: Regret Upper Bound

Theorem

The expected cumulative regret is upper bounded by,

$$\sum_{t=1}^{T} \mathbb{E}[R(t)] \leq K\Delta_{\max} \sum_{\tau=1}^{t} \exp\left(\frac{-\tau\Delta_{\min}^2}{2B^2}\right) \tag{6}$$

Which is a constant

- This bound relies on the distribution being unbiased and well sampled
- Well sampled meaning the number of samples associated with each $\tilde{p}_i(t)$ should be large
- In the absence of the invertibility assumptions, even logarithmic regret is not guaranteed

Pure Exploration Framework

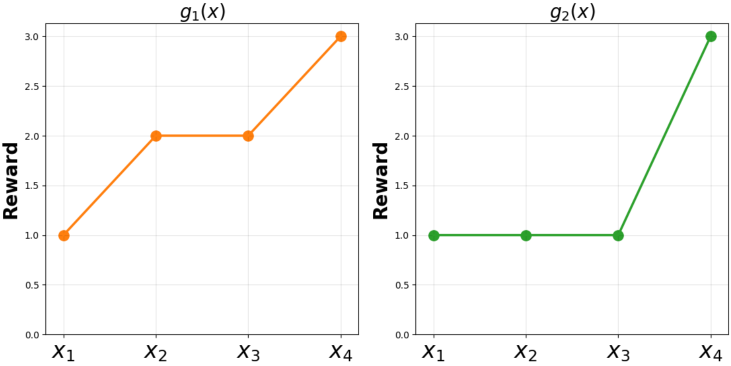
- Heavy restrictions required on allowed Bandit Instances to use method for regret minimization
- Regret minimization has exploration-exploitation trade-off
- Cannot actively learn distribution
- Look towards other frameworks - Pure Exploration Setting

PAC Algorithm for Correlated Bandits

- Given a set of arms with known reward functions $g_1, \ldots, g_K$, and an underlying latent random variable X with support points $X = \{x_1, ..., x_n\}$. The goal is to find the best arm
- We propose a $(0, \delta)$-PAC Algorithm for a correlated bandit based on RRPULL + PIEST algorithm by Gupta et al. (2018) [1] and the Successive Elimination algorithm (E Even-Dar et al. 2002 [4])
- The former learns distributions of rewards from indirect samples

Preprocessing

- Merge into 'superpoints'. Here x_2 and x_3 can be merged
- Normalise all rewards to [0,1], for ease of analysis



Notation

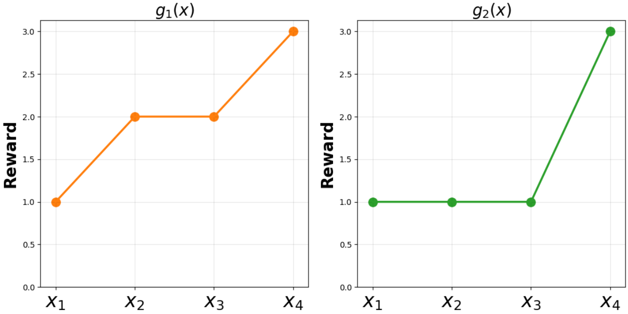
Borrowing notation from Gupta et al. (2018) [1], we define the following.

- As before,
  The true distribution of X is $P_X = [p_1, p_2, ..., p_n]^T$
  Our best estimate of P_X is $\tilde{P}_X = [\tilde{p}_1(t), \tilde{p}_2(t), ..., \tilde{p}_n(t)]^T$
- $\{z_{k,1}, ..., z_{k,m_k}\}$ - set of possible outcomes of the function g_k where m_k is the number of distinct outputs of g_k
- Sample Generation Matrix A_k of size $m_k \times n$

$$A_k(i,j) = \begin{cases} 1 & g_k(x_j) = z_{k,i} \\ 0 & \text{otherwise.} \end{cases} \tag{7}$$

- $A = [A_1^T, A_2^T, ..., A_K^T]$ of size $m \times n$ where $m = m_1 + m_2 + ... + m_K$
- A is the combined matrix and it captures information about the entire instance

Notation



$$A = \left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \hline 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\text{becomes}} A = \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ \hline 1 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] \tag{8}$$

Notation

- $q_{k,i}$ is the probability of observing $z_{k,i}$ each time arm k is pulled; $Q = [q_{1,1}, ..., q_{1,m_1}, ..., q_{K,m_K}]^T$, $\hat{Q}$ is the empirical estimate of Q
- Matrix A relates the entries of P_X and Q

$$q_{k,i} = \sum_{j=1}^{n} A_k(i,j) p_j \tag{9}$$

$$AP_X = Q \implies P_X = A^+ Q \tag{10}$$

- Lastly, as before, we have

$$\tilde{g}_s^t = \sum_{j=1}^{n} g_s(x_j) \tilde{p}_j(t) \tag{11}$$

Algorithm

Algorithm 2 Successive Elimination for Correlated Bandits

Input: Alphabet $\{x_1, ..., x_n\}$, Functions $\{g_1, ..., g_K\}$, Set of arms S
Initialize: $t = 0, t_k = 0 \; \forall \; k, \; t_{k,i} = 0 \; \forall \; i, k, \;\; \tilde{p}_j(0) = \frac{1}{n} \; \forall \; j$
while $|S| > 1$ **do**
  Preprocess by merging into superpoints
  Update the matrix A based on S and reduced alphabet
  Pull arm $s_t = mod(t, |S|) + 1$, observe output y_t
  $t_{s_t} = t_{s_t} + 1, \; t = t + 1$
  if $y_t = z_{s_t,i}$ **then**
    $t_{s_t,i} = t_{s_t,i} + 1$
  end if
  $\hat{q}_{s,i} = \frac{t_{s,i}}{t_s} \; \forall \; i, s$
  Obtain estimates $\tilde{p}_j(t)$ as $\tilde{P}_X = A^+ \hat{Q}$
  Let $\tilde{g}^t_{max} = \max_{s \in S} \tilde{g}^t_s, \; \alpha_t = \sqrt{\frac{\log(cKt^2/\delta)}{t}}$
  For every arm $s \in S$ s.t $\tilde{g}^t_{max} - \tilde{g}^t_s \geq 2\alpha_t$. Set $S = S \backslash s$
end while

Results Needed for Analysis

Theorem (Gupta et al., 2018 [1], Theorem 1)

It is possible to achieve asymptotically consistent estimation of probability distribution of X if and only if $rank(A) = n$.

Theorem (Gupta et al., 2018 [1], Theorem 6)

It is possible to achieve estimation error of probability distribution of X of $O(\frac{1}{t})$ if $rank(A) = n$.

The preprocessing step ensures that all instances have $rank(A) = n$.

Results Needed for Analysis

Theorem

The empirical estimate of the distribution P_X, $\tilde{P}_X = A^+\hat{Q}$ is unbiased

Proof:
By the definition of $\hat{q}_{s,i}$, $\hat{Q}$ is unbiased. Therefore, $\mathbb{E}[\hat{Q}] = Q$. Hence,

$$\mathbb{E}[\tilde{P}_X] = \mathbb{E}[A^+\hat{Q}] = A^+\mathbb{E}[\hat{Q}] = A^+Q = P_X \tag{12}$$

Theorem

The Successive Elimination for Correlated Bandits is a $(0, \delta)$-PAC algorithm, and with probability $(1 - \delta)$ its arm complexity is bounded by $O\left(\frac{\log(K/\delta\Delta_{min})}{\Delta_{min}^2}\right)$

Proof of $(0, \delta)$-PAC

For any time t and action $s \in S_t$, we have,

$$Pr[|\tilde{g}_s^t - \mu_s| \geq \alpha_t] \leq \exp^{-\alpha_t^2 t} \leq \frac{\delta}{cKt^2} \tag{13}$$

because the Hoeffding inequality can be applied to $\tilde{g}_s^t$, being an unbiased estimate of μ_s from theorem [6].

With probability at least $(1 - \frac{\delta}{K})$ for any time t and action $s \in S_t$, $|\tilde{g}_s^t - \mu_s| \leq \alpha_t$.

Hence, with probability $(1 - \delta)$, best arm is never eliminated since as $\alpha_t \to 0$ as t increases, eventually every non-best arm is eliminated.

Sample Complexity

To eliminate a non-best arm s_i, we need to reach a time t_i such that,

$$\hat{\Delta}_{t_i} = \tilde{g}_{s^*}^{t_i} - \tilde{g}_{s_i}^{t_i} \geq 2\alpha_{t_i} \tag{14}$$

Definition of α_t combined with the assumption that $|\tilde{g}_s^t - \mu_s| \leq \alpha_t$ yields,

$$\Delta_i - 2\alpha_t = (\mu_{s^*}(X) - \alpha_t) - (\mu_{s_i}(X) + \alpha_t) \geq \tilde{g}_{s^*}(X) - \tilde{g}_{s_i}(X) \geq 2\alpha_t \tag{15}$$

which holds with probability atleast $(1 - \frac{\delta}{K})$ for

$$t_i = O\left(\frac{\log(K/\delta\Delta_i)}{\Delta_i^2}\right) \tag{16}$$

The last non-best arm will thus be eliminated and the best arm output at

$$t = O\left(\frac{\log(K/\delta\Delta_{min})}{\Delta_{min}^2}\right) \tag{17}$$

Questions?

References

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Thank you!