A Comparison between Extended and Unscented Kalman Filter in Non-Linear Estimation

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I. Introduction

The simplicity of the Kalman Filter in solving linear sequence estimation problems has made it the tool of choice for Engineers for the past many decades. However the Kalman Filter algorithm is only applicable to the estimation of signals that follow a linear state space model.
In Control Engineering, more often than not, we need to deal with systems that do not follow a linear model, but instead can be described through non linear functions. For instance, a popular application of the Kalman Filter is in sensor fusion for Autonomous Vehicle Navigation. As an autonomous vehicle moves along its trajectory, it constantly receives complementary information from numerous sensors such as RADAR and LIDAR. The job of the filtering algorithm is to fuse the incoming information to make well informed decisions about the steering angle. Combining this information will involve vector calculations which in turn invariably involve non-linear trigonometric functions.
Soon after the introduction of the Kalman Filter in 1961, an Extended Kalman Filter (EKF) was proposed. The EKF solves the non-linear estimation problem by locally linearizing the State Transition and Measurement functions. This technique proves to be useful for many applications that can be modelled as being approximately linear. Moreover, the Taylor series becomes a more accurate local estimate of a function if the update frequency is high. Despite its merits, the EKF suffers from many issues which are discussed later.
To overcome the drawbacks posed by the EKF, Julier and Uhlmann introduced the Unscented Kalman Filter (UKF) in [1]. The UKF overcomes many of the challenges that the EKF fails at and is overall more robust in the face of more intense non-linearities. In this paper, we compare the merits of the these two approaches to non-linear filtering problems.

II. Preliminaries and Notation

The linear discrete time invariant state space model is usually written as,

$$x_{k+1} = F_k x_k + G_k u_k \tag{1}$$
$$y_k = H_k x_k + v_k \tag{2}$$

Under a general non-linear assumption, these equations will become

$$x_{k+1} = f(x_k, k) + G u_k \tag{3}$$
$$y_k = h(x_k, k) + v_k \tag{4}$$

Here $f(), h()$ can in general be any time dependent non-linear functions, however in this paper, we only consider the case of non-linear functions that are constant with time.
In this notation, y_k are the observable vectors and x_k is the unknown underlying state of the system. As usual, w_k and v_k are the process noise and measurement noise respectively. Usual assumptions such as noise terms being zero mean and uncorrelated apply.

III. Extended Kalman Filter

The Extended Kalman Filter (EKF) uses the linear Taylor Series approximation of a function to transform it into a time variant linear state space model.
To derive the linear model, first we must find the Jacobian matrices of the functions $f(), h()$

$$F_k = \frac{\partial f(\mathbf{x}, t)}{\partial \mathbf{x}} \tag{5}$$
$$H_k = \frac{\partial h(\mathbf{x}, t)}{\partial \mathbf{x}} \tag{6}$$

Once the Jacobian matrices of (5) and (6) have been derived analytically or numerically, the non-linear state space equation can be linearized with the state and observation variables as the error terms
$\delta_k = x_k - x_k^R$ and $z_k = y_k - h(x_k^R)$, where x_k^R is the reference trajectory the state space takes in absence of any external disturbances. Once a linear state space model is available, the problem reduces to a usual Kalman Filtering problem.

The use of the EKF has two well known drawbacks.

1) Linearisation can produce a highly unstable filter if assumptions of local linearity are violated.
2) The derivation of the Jacobian Matrices is non trivial and leads to implementation difficulties. Moreover, the Jacobian may be expensive to derive numerically and may not even exist in case of non-differentiable functions.

These drawbacks are resolved to some extent by using the Unscented Kalman Filter which is described next.

IV. Unscented Kalman Filter

The Unscented Kalman Filter (UKF) makes uses of the principle that a set of discretely sampled points can be used to parameterize mean and co-variance.
In the UKF, the distribution of every state is specified using

a minimal set of deterministically chosen points called **sigma points**. To choose these sample points the UKF takes the help of the **unscented transform**.
The unscented transform is a technique used to approximate the distribution of a random variable undergoing a Non-Linear transformation. The procedure for it goes as follows:

1) Choose a set of $2n+1$ Sigma Points, such that their sample mean is the true mean of the state $\bar{x}$, and the sample covariance is the error covariance P_{xx}. Here n is the dimention of the state space. In particular the primary reference [1] provides a deterministic way to get these points.
2) Apply the non-linear function to each sigma point to yield a cloud of transformed points with sample mean $\bar{y}y$ and covariance P_{yy}. This is ensured by weighting each point y_i by a specially designed weight $\mathcal{W}_i$. That is, we form the equations:

$$\bar{y} = \sum_{i=0}^{2n} \mathcal{W}_i y_i \tag{7}$$

$$P_{yy} = \sum_{i=0}^{2n} \mathcal{W}_i (y_i - \bar{y})(y_i - \bar{y})^T \tag{8}$$

Once the 1^{st} and 2^{nd} order statistics for the transformed points (y's) are known, the usual innovations recursion procedure can be follow to apply Kalman Filtering.
Desirable properties of this transform, especially when compared to the EKF are the following,

- The mean is approximated to a higher order of accuracy than the EKF
- Through this transform, we approximate the distribution of the state variable x rather than the non-linear function. This allows information from higher orders to be accommodated in the transform.
- The computation or even existence of the Jacobian Matrices is not required.

In the next section we compare the performance of the EKF and the UKF on an aircraft re-entry problem to recreate the results from the reference [1].

V. An aircraft Re-Entry Problem

Consider the problem of tracking the trajectory of an aircraft as it performs atmospheric re-entry from a very high altitude and at a very high speed. A radar shown in figure 1 as a circle, relays information to the ground-station. The forces acting on the object are atmospheric-drag, gravitation and random buffeting accelerations which are modelled as noise.
As can be seen from the simulated trajectory of figure 1, initially the motion of the vehicle is straight-line ballistic, but eventually, because of drag, the motion becomes almost vertical.

To illustrate the merits of the EKF and the UKF, we will formulate the tracking problem as a non-linear estimation problem. The state **x** of the vehicle contains the following components:

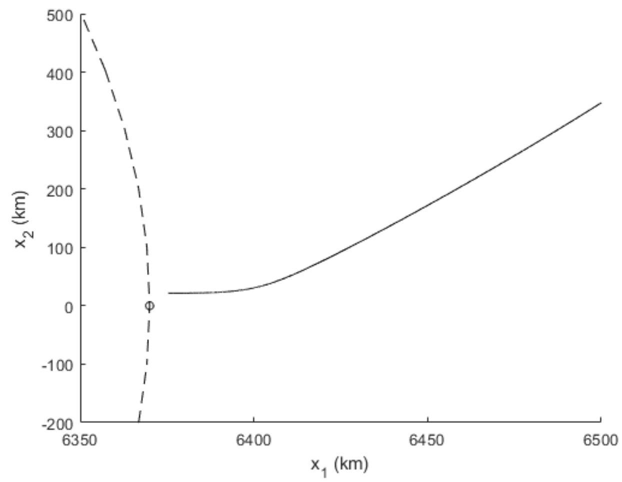

Figure 1: An example trajectory followed on re-entry

- x_1 and x_2 are the x and y coordinate of the vehicle with respect to the grid shown in figure 1
- x_3 and x_4 are the x and y components of the vehicles velocity
- x_5 Represents a constant aerodynamic property of the vehicle. This constant is unknown at the start and is estimated throughout.

The differential equations associated with the model are as below,

$$\dot{x_1} = x_3(k) \tag{9}$$
$$\dot{x_2} = x_4(k) \tag{10}$$
$$\dot{x_3} = D(k)x_3(k) + G(k)x_1(k) + v_1(k) \tag{11}$$
$$\dot{x_3} = D(k)x_4(k) + G(k)x_2(k) + v_2(k) \tag{12}$$
$$\dot{x_5} = v_3(k) \tag{13}$$
$$\tag{14}$$

Here, $D(k)$ is the drag force term, $G(k)$ is the gravity term, $v_i(k)$ are the process noise terms. Defining $R(k) = \sqrt{x_1^2(k) + x_2^2(k)}$ as distance from Earth's center and the total speed$V(k) = \sqrt{x_3^2(k) + x_4^2(k)}$, we can write,

$$D(k) = \beta(k) \exp\left(\frac{R_o - R(k)}{Ho}\right) V(k) \tag{15}$$

$$G(k) = -\frac{Gm_o}{R^3(k)} \tag{16}$$

$$\beta(k) = \beta_o \exp(x_5(k)) \tag{17}$$

The values of all constants with 'o' subscript were taken from the reference [1]. Further all state and covariance matrix initializations were kept the same as in the reference paper. Their values can be seen in the code files under the code/vehicle folder.

The measurements from the radar are in the form of a range r and a bearing angle θ, equations for them are given by,

$$r(k) = \sqrt{(x_1(k) - x_r)^2 + (x_2(k) - y_r)^2} + w_1(k) \tag{18}$$

$$\theta(k) = \tan^{-1}\left(\frac{x_2(k) - y_r}{x_1(k) - x_r}\right) + w_2(k) \tag{19}$$

Here, $w_1(k)$ and $w_2(k)$ are measurement noise terms that arise from random disturbances in the radar receiver.
Based on the above formulation, the ODEs were discretized using Euler's Scheme. MATLAB tools were then used to apply Kalman Filtering to this state tracking problem.
The results from applying the EKF and the UKF to this problem were computed and are shown in the following figures.

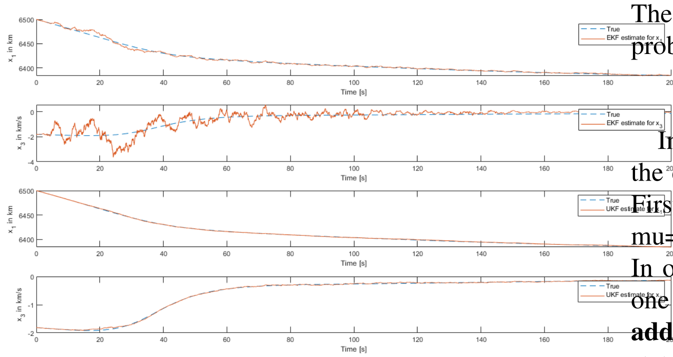

Figure 2: Tracking the states - x_1 and x_3 by EKF (top 2 cells) and UKF (bottom 2 cells). For a clearer picture please see images under results folder

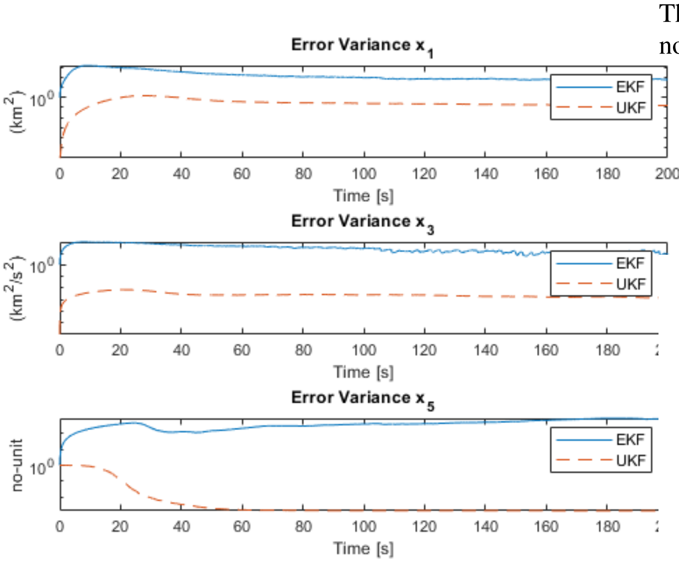

Figure 3: Collated plots for error variance values over time

In figure 2 we see that the UKF does a much better job of following the ground truth state of the vehicle. The EKF approximation is unsteady and has large error and error variance, whereas the UKF performance is almost indistinguishable from the true trajectory.
Consistent with the analysis of the aircraft problem present in the paper [1], the figure 3 illustrates that the error covariance of the EKF is much higher than that of the UKF, further, the error covariance of the constant term x_5 never decreases leading to a poor and biased estimate of the constant vehicle parameter x_5.
The code files to generate these results can be found under code/vehicle. Specifically the **main_vehicle.m** file.

Conclusions

From the aircraft tracking problem, we can conclude that, in the presence of many layers of non-linearity, The UKF can track the state of the vehicle quite well.
The EKF on the other hand is quite unsuccessful at the problem.

VI. Effect of Extent of Non-Linearity

In this section we perform some new experiments to explore the effect of increasing the extent of non-linearity in a model. First we consider the standard van der Pol (vdp) oscillator with mu=1. This oscillator is specified by the ODEs (20) and (21). In our estimation model we take the measurement process as one of the states itself. The noise model considered is **non-additive** and is instead proportional to the magnitude of the state being considered.
This noise model is quite realistic for measurement processes since errors are often specified as a percentage of the measured quantity, and this exactly fits into the model considered in (22). Further the formulation of the EKF and UKF quite naturally extends to non-additive noise.
The MATLAB tool-box has a method of specifying non-linear noise.

$$\dot{x_1}(k) = x_2(k) \tag{20}$$

$$\dot{x_2}(k) = (1 - x_1^2(k))x_2(k) - x_1(k) \tag{21}$$

$$y(k) = x_1(k)(1 + v[k]) \tag{22}$$

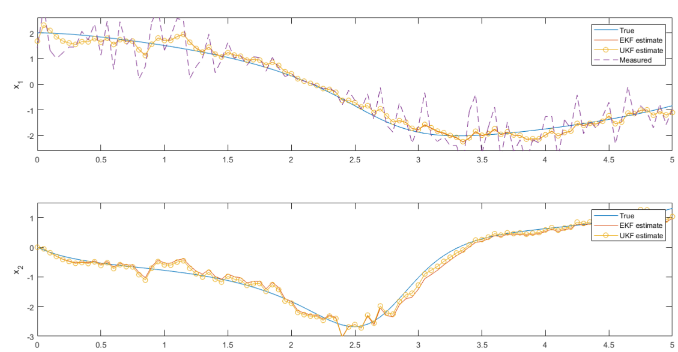

Figure 4: Tracking of the 2 states by the EKF and UKF, we have state x_1 above and state x_2 below

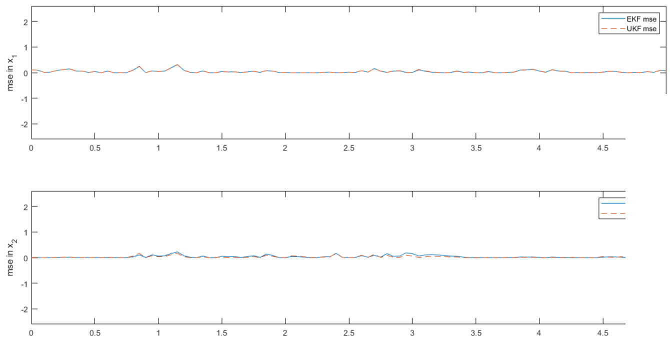

Figure 5: Mean squared error in estimating the 2 states, x_1 above and x_2 below. Please see results folder for clearer picture

The non-linearity in this model is not much and we see that both EKF and UKF have almost identical performance. This can be seen from both the tracking over time in figure 4 and from the mean-squared error in state estimation over time figure 5.

Modified VDP Oscillator

Next, to push the ability of the estimators, consider a modified model with an additional term dependent on $x_2(k)$ added to measurement process.

$$\dot{x_1}(k) = x_2(k) \tag{23}$$

$$\dot{x_2}(k) = (1 - x_1^2(k))x_2(k) - x_1(k) \tag{24}$$

$$y(k) = x_1(k)(1 + v[k]) + \sin(x_2(k)) \tag{25}$$

When the same experiments as the usual vdp oscillator are applied to this model, we get the results as shown in figure 6 and 7

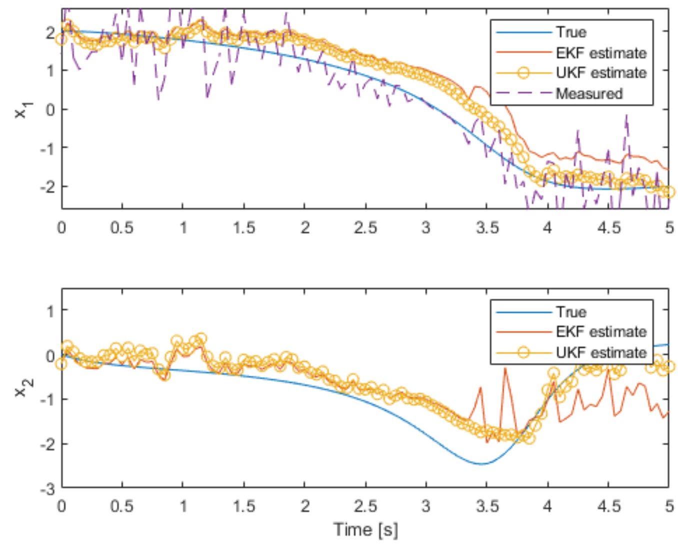

Figure 6: Tracking of the 2 states

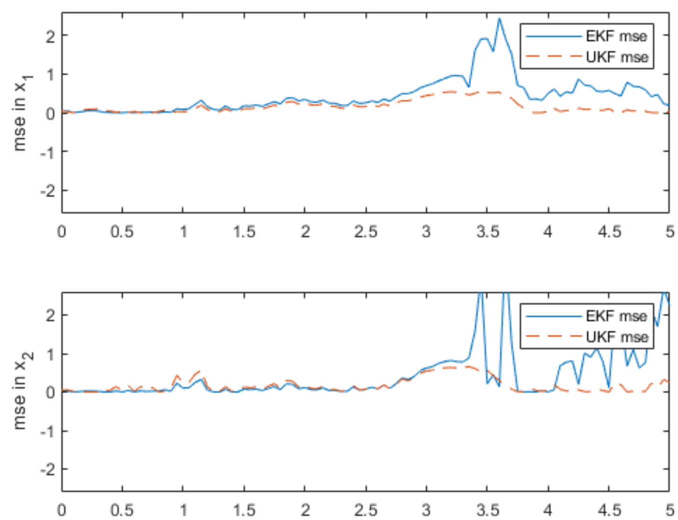

Figure 7: Mean squared error in estimating the 2 states

VII. Conclusions

From these analytical experiments we see that, the ability of the EKF in handling non-linearity is quite limited. Whenever there are non polynomial terms in either the state transition function or the measurement functions they will have an infinite Taylor series expansion. Often, it is a grossly poor approximation to arbitrarily terminate this expansion at the linear level. The poor approximation leads to poor results.
Likewise, the performance of the UKF also suffers and it is not able to accurately track the state trajectory. But the performance is still far superior to the EKF.
One of the ways in which this superiority can be seen is that the EKF has a jittery and peaky filter output whereas the UKF has a smooth output although it does not track the state exactly.

References

[1] S.J.Julier and J.K.Uhlmann. A New Extension of the Kalman Filter to Non linear Systems. In Proc. of AeroSense: The 11th Int. Symp. on Aerospace/Defence Sensing, Simulation and Controls., 1997

[2] MathWorks tutorial on Non linear state estimation. Link: https://in.mathworks.com/help/control/ug/nonlinear-state-estimation-using-unscented-kalman-filter.html